An Implicit Lambda Scheme

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This paper is concerned with the numerical simulation of transonic inviscid flows. The unsteady Euler equations are written in characteristic-type variables, discretized in time, and solved by means of an implicit technique or an alternating direction implicit technique, for one- and two-dimensional flows, respectively, so that only block-tridiagonal systems have to be solved at each time level. The present method has been successfully applied to one-dimensional transonic internal flows and two-dimensional transonic external flows within the framework of thin airfoil theory, and can be generalized to transonic two- and three-dimensional flows for arbitrary geometries. In all computed cases the present technique has proved itself reliable and about one order of magnitude more efficient than its well-known explicit counterpart.

Introduction

THE aerodynamic design of high-speed aircraft is very difficult because of the possible presence of unsteady shock motion. These waves move within the flowfield and produce large variations in the aerodynamic forces and moments acting on the body under consideration. Although the ultimate goal of aerodynamic design at transonic speeds is to produce shock-free shapes, this is at best achieved at design conditions, whereas for most off-design operations, steady or unsteady shock waves are present in the flowfield. Moreover, the interaction of such shock waves with the boundary layer near the body surface is likely to produce flow separation and therefore large aerodynamic losses and low efficiencies.

Before the advent of high-speed, large-storage computers, the development of advanced aircrafts relied upon the designer's experience and an extensive use of wind-tunnel testing on models. More recently, due to the remarkable progress in the field of computational fluid dynamics, computer simulation has become a very useful tool in the prediction of complex flowfields. However, even though the computer is considered by some[1] as the alternative, cheaper "wind tunnel" of the future, much progress must still be made before one can rely upon computational fluid dynamics as the primary aerodynamic design tool. In fact, the solution of transonic, viscous, turbulent, unsteady, three-dimensional flow problems, although feasible in principle, is as yet impractical and certainly uneconomical. In spite of the fact that complex and challenging unsteady high-speed viscous flow problems have been successfully solved by means of well-established numerical techniques,[2-6] the solution of three-dimensional transonic flow around an aircraft would require prohibitive computation costs, even for the simpler case of inviscid flow.

As a matter of fact, a reliable and efficient numerical technique for solving unsteady transonic inviscid flows is not yet available. Although well-established explicit techniques[7] have a well-known stability limitation on the integration time step, which strongly reduces their efficiency, more modern and sophisticated implicit methods[8-13] are thus far limited to the case of steady flows. In particular, these fast solvers of the full potential or transonic small-perturbation equations (which are based on various factorization techniques[8-10] or on the multigrid approach[12]) have, in spite of their success in reducing computation costs as a result of their remarkable convergence rates, several drawbacks. For example the shock is usually slightly misplaced, more than one numerical solution can be obtained for the same physical problem (see, e.g., Ref. 13), and numerical "tricks" such as artificial viscosity or compressibility have to be explicitly introduced into the numerical scheme in order to capture the shock waves present in the flowfield. For these reasons, even for the case of steady flow problems, many researchers still prefer time-dependent explicit techniques, which, although much slower due to their stability restriction on the time step, are more reliable and physically sounder.[14-16]

In particular, quite recently, an old idea has been developed and implemented by Moretti and other researchers[17-20] to produce the mathematically attractive and physically meaningful "lambda scheme." Such a technique writes the inviscid compressible conservation law equations in characteristic-type (Riemann-type) variables. Therefore, for each dependent variable, the direction of perturbation-propagation (characteristic line) is known and windward differencing is used, consistent with the physics of the problem. Moreover, for the weak shocks typical of transonic regimes, one of the variables is continuous across the shocks, which are therefore captured by the numerical technique without any numerical "tricks" such as artificial viscosity or compressibility (the method obviously has a natural numerical viscosity associated with its use of windward differences). A further noteworthy advantage of the lambda scheme is related to the concept of "nonreflecting boundary conditions"[21-23]: by imposing characteristic-type numerical boundary conditions, it is possible to allow for disturbances that reach the free boundary of the integration domain to exit without being improperly reflected by it. It is then obvious that such boundary conditions can be straightforwardly implemented when the difference equations are already written in characteristic-type variables. Finally, the method can be naturally extended to three-dimensional flows.[20] The major drawback of such a technique is its computational inefficiency, typical of all explicit methods.

As outlined above, all of the appealing features of the lambda scheme are associated with its use of the characteristic-type dependent variables and its main limitation is due to the numerical method used thus far to discretize and integrate the governing equations. Therefore, it appears very promising and worthwhile to develop an "implicit lambda scheme." Such a new method would retain all of the physically sound features of its explicit predecessor, while removing its stability limitation on the integration time step. Moreover, it could be employed as a true time-dependent technique for unsteady flows for which the time step would be

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limited by accuracy requirements, as well as a steady-state solver for which the time step would become a relaxation parameter. In the latter case an even faster convergence rate could be achieved by means of any available factorization or multigrid method (see, e.g., Refs. 8, 10, and 12).

For the case of a true time-dependent approach of primary interest here, an implicit method is the obvious choice for the case of one-dimensional flows, whereas an alternating direction implicit (ADI) technique appears to be the most attractive approach for the case of two-dimensional flows. In the last decade, reliable implicit and "linearized block implicit" numerical techniques, derived from the ADI methods of Douglas and Gunn[24] or of Peaceman and Rachford,[25] have been developed by several researchers for viscous flow problems and, in particular, for the time-dependent compressible Navier-Stokes equations.[2,3,26] These methods have the very desirable features of being (unconditionally) stable and of requiring the solution of only block-tridiagonal systems, for which a very fast and reliable Gaussian reduction algorithm is available.[27,30]

In the present paper a reliable and efficient implicit lambda scheme is developed for the cases of one- and two-dimensional flows within the framework of the thin airfoil theory. An incremental (delta) form of a fully implicit technique and of the Douglas and Gunn ADI method[24,28] is applied to the governing equations of the classical explicit lambda scheme[17-20] for the cases of one- and two-dimensional flows, respectively. The numerical techniques are described in the following sections and their reliability and efficiency are then demonstrated by means of a few sample calculations.

One-Dimensional Flows

Numerical Method

The Euler equations for the case of one-dimensional compressible isentropic flow inside a variable area duct are given in dimensionless characteristic-type (Riemann) variables[20] as

$$C_\tau + D_\tau = -\lambda_+ C_x - \lambda_- D_x \tag{1}$$

$$C_\tau - D_\tau = -\lambda_+ C_x + \lambda_- D_x - \alpha' (C+D)(C-D)(\gamma-1)/4 \tag{2}$$

In Eqs. (1) and (2) x and τ are the dimensionless longitudinal coordinate and time (i.e., the independent variables), α' is the specific rate of change of the cross-sectional area W $\{\alpha' = (\mathrm{d}W/\mathrm{d}x)/W\}$, C and D are the two Riemann variables, and λ_+ and λ_- are the slopes of their characteristic lines. These last four variables are related to the nondimensional speeds of fluid U and sound A and to the specific heats ratio γ by the following equations:

$$C = U + 2A/(\gamma-1) \tag{3}$$

$$D = U - 2A/(\gamma-1) \tag{4}$$

$$\lambda_\pm = U \pm A \tag{5a,b}$$

The main advantage of using Eqs. (1) and (2), i.e., the Euler equations written in characteristic-type variables, is that each term of the type $\lambda_+ C_x$ relates to the effect of the propagation of a given wave along its characteristic line. Therefore, its finite difference representation has to be windward with respect to the direction of propagation of the wave signal, so that the numerical method is automatically consistent with the physics of the problem.

In the present approach Eqs. (1) and (2) are written in delta form and linearized in time to give

$$\Delta C/\Delta\tau + \Delta D/\Delta\tau = -\lambda_+^n \Delta C_x - C_x^n \Delta\lambda_+ - \lambda_+^n C_x^n$$
$$- \lambda_-^n \Delta D_x - D_x^n \Delta\lambda_- - \lambda_-^n D_x^n \tag{6}$$

and

$$\Delta C/\Delta\tau - \Delta D/\Delta\tau = -\lambda_+^n \Delta C_x - C_x^n \Delta\lambda_+ - \lambda_+^n C_x^n$$
$$+ \lambda_-^n \Delta D_x + D_x^n \Delta\lambda_- + \lambda_-^n D_x^n - \alpha' \{ (C+D)^n \Delta(C-D)$$
$$+ (C-D)^n \Delta(C+D) + (C+D)^n (C-D)^n \} (\gamma-1)/4 \tag{7}$$

where $\Delta C = C^{n+1} - C^n$, $C^n = C(x,t^n)$, $C^{n+1} = C(x,t^n + \Delta t)$, etc. Equations (6) and (7) are then discretized in space by taking all of the old time level derivatives as three-point second-order-accurate windward differences and the unknown derivatives of the incremental variables as two-point first-order-accurate windward differences, that is,

$$C_x^n = (3C_i^n - 4C_{i-1}^n + C_{i-2}^n)/2\Delta x \tag{8}$$

$$D_x^n = s(3D_i^n - 4D_{i-s}^n + D_{i-2s}^n)/2\Delta x \tag{9}$$

$$\Delta C_x = (\Delta C_i - \Delta C_{i-1})/\Delta x \tag{10}$$

$$\Delta D_x = s(\Delta D_i - \Delta D_{i-s})/\Delta x \tag{11}$$

where $s=1$ at the supersonic flow grid points and $s=-1$ at the subsonic flow grid points. By introducing Eqs. (8-11) into Eqs. (6) and (7), a linear 2×2 block-tridiagonal system is obtained for the unknowns ΔC_i and ΔD_i $(i=1,2,\ldots,N)$, which is easily solved by standard block-tridiagonal Gaussian reduction.[27]

The solution at the new time level is then obtained as

$$C_i^{n+1} = C_i^n + \Delta C_i \tag{12}$$

$$D_i^{n+1} = D_i^n + \Delta D_i \tag{13}$$

It is noteworthy to point out that, whereas the method is only first-order accurate in time, if a steady-state solution is sought as the asymptotic decay of a transient phenomenon, the final solution will be second-order accurate. Also, at the grid points immediately adjacent to the boundaries it is not always possible to use three-point windward differences, and standard second-order-accurate central differences are then used at these locations when necessary. The resulting local violation of the physics of the problem is usually of no consequence unless it involves a differencing of D across a shock. In such a case, D being sharply discontinuous across the shock, a two-point first-order-accurate difference is to be used for D_x^n in order to avoid an otherwise catastrophic numerical behavior.

Boundary Conditions

At the inlet boundary, the flow is always considered to be subsonic and the total temperature is assumed to be equal to its upstream value, i.e.,

$$(\gamma+1)C^2 + 2(3-\gamma)CD + (\gamma+1)D^2 = 16A_0^2/(\gamma-1) \tag{14}$$

where A_0 is the prescribed nondimensional upstream stagnation speed of sound. Equation (14) is then linearized and written in incremental form, to give

$$8\lambda_+^n \Delta C + 8\lambda_-^n \Delta D = 16A_0^2/(\gamma-1) - (\gamma+1)(C^{n^2} + D^{n^2})$$
$$-2(3-\gamma)C^n D^n \tag{15}$$

which is the only inlet boundary condition and is coupled with the left-running (D) wave equation,

$$D_\tau + \lambda_- D_x = \alpha' UA \tag{16}$$

Equation (16) is discretized in time, linearized, and solved in the same way as the governing equations.

At the outlet boundary the subsonic flow as well as the supersonic flow cases have been considered. In the latter case, no boundary condition is needed insofar as the C and D waves both reach the outlet section, which is then treated as an internal point. For the subsonic flow case, the C wave still reaches the outlet section and the right-running (C) wave equation is thus applicable,

$$C_\tau + \lambda_+ C_x = -\alpha' UA \tag{17}$$

Moreover, the outlet static pressure $p(\tau)$ must be prescribed. For the present case of isentropic flow, the outlet speed of sound $A_u(\tau)$ is thus known, insofar as it is related to $p(\tau)$ by the relationship,

$$A_u(\tau) = \sqrt{\gamma}\, p(\tau)^{(\gamma-1)/2\gamma} \tag{18}$$

From Eqs. (3), (4), and (18) the required boundary condition is easily obtained as

$$(C-D)[(\gamma-1)/4] = A_u(\tau) \tag{19}$$

Equation (19), like all other boundary conditions, is obviously written in delta form and treated numerically like the governing equations.

Two-Dimensional Flows

Numerical Method

The Euler equations for two-dimensional inviscid isentropic compressible flows are given in dimensionless bicharacteristic-type variables[20] as

$$C_\tau + (U+A)C_x + VC_y = -AV_y \tag{20}$$

$$D_\tau + (U-A)D_x + VD_y = AV_y \tag{21}$$

$$E_\tau + UE_x + (V+A)E_y = -AU_x \tag{22}$$

$$F_\tau + UF_x + (V-A)F_y = AU_x \tag{23}$$

where x and y are the nondimensional longitudinal and vertical coordinates, τ the time, U and V the longitudinal and vertical components of the velocity of the fluid, A the speed of sound, and C, D, E, and F the four bicharacteristic dependent variables, given by

$$C = U + 2A/(\gamma-1) \tag{24}$$

$$D = U - 2A/(\gamma-1) \tag{25}$$

$$E = V + 2A/(\gamma-1) \tag{26}$$

$$F = V - 2A/(\gamma-1) \tag{27}$$

The governing equations (20-23) are then replaced by the following linear combinations of the same[20]:

$$C_\tau + D_\tau = -\{(U+A)C_x + VC_y + (U-A)D_x + VD_y\} \tag{28}$$

$$E_\tau + F_\tau = -\{UE_x + (V+A)E_y + UF_x + (V-A)F_y\} \tag{29}$$

$$C_\tau - D_\tau + E_\tau - F_\tau = -2\{(U+A)C_x - (U-A)D_x + (V+A)E_y - (V-A)F_y\} \tag{30}$$

$$C - D - E + F = 0 \tag{31}$$

For the sake of conciseness, from now on all of the algebra will be carried on only for Eq. (28) as an example, the other equations being treated in exactly the same manner, with the exception of Eq. (31), which is used to eliminate F in favor of C, D, and E at all stages of the numerical procedure. The governing equations (28-31) are linearized and written in delta form, e.g.,

$$\begin{aligned}\Delta C/\Delta\tau + \Delta D/\Delta\tau = -\{&(U+A)^n\Delta C_x + C_x^n\Delta(U+A) \\ &+ (U+A)^n C_x^n + V^n\Delta C_y + C_y^n\Delta V + V^n C_y^n + (U-A)^n\Delta D_x \\ &+ D_x^n\Delta(U-A) + (U-A)^n D_x^n + V^n\Delta D_y + D_y^n\Delta V + V^n D_y^n\}\end{aligned} \tag{32}$$

In order to retain the block-tridiagonal nature of the algebraic systems to be solved, the Douglas and Gunn ADI procedure[24] is applied to the linearized incremental equations. At the first sweep of the ADI technique ($\sim$sweep), Eq. (32) thus becomes

$$\begin{aligned}&\tilde\Delta C/\Delta\tau + \tilde\Delta D/\Delta\tau + (U+A)^n\tilde\Delta C_x + C_x^n\tilde\Delta(U+A) \\ &+ (U-A)^n\tilde\Delta D_x + D_x^n\tilde\Delta(U-A) = -\{(U+A)^n C_x^n \\ &+ V^n C_y^n + (U-A)^n D_x^n + V^n D_y^n\}\end{aligned} \tag{33}$$

It is to be noted that Eq. (33) is obtained by evaluating all the x derivatives at the new time level (implicitly) and all the y derivatives at the old time level (explicitly) in Eq. (32); moreover, the source-like terms of the type $C_x^n\Delta(U+A)$ and $C_y^n\Delta V$ have been treated exactly as those arising from the linearization of the same nonlinear terms, $(U+A)^n\Delta C_x$ and $V^n\Delta C_y$, respectively. This is a somewhat arbitrary choice, insofar as an implicit treatment of all the source-like terms would also produce block-tridiagonal systems. The present choice, however, appears to be justified in the spirit of numerically treating all physical waves in the most consistent manner and has been slightly more efficient in some of the calculations presented later in this paper. Finally, it is noteworthy that the right-hand side of Eq. (33) is the steady-state form of the corresponding governing equation (28), a well-known peculiarity of the delta approach.

At the second and final sweep of the ADI technique, Eq. (32) is written as

$$\begin{aligned}&\bar\Delta C/\Delta\tau + \bar\Delta D/\Delta\tau + V^n\bar\Delta C_y + C_y^n\bar\Delta V + V^n\bar\Delta D_y + D_y^n\bar\Delta V \\ &= -\{(U+A)^n\tilde\Delta C_x + C_x^n\tilde\Delta(U+A) + (U+A)^n C_x^n \\ &+ C_y^n V^n + (U-A)^n\tilde\Delta D_x + D_x^n\tilde\Delta(U-A) \\ &+ (U-A)^n D_x^n + D_y^n V^n\}\end{aligned} \tag{34}$$

where all of the y derivatives and the associated source-like terms are now treated implicitly at the final ($-$) level whereas all of the x derivatives and associated source-like terms are evaluated explicitly at the "predictor" ($\sim$) level. For coding convenience the second sweep [Eq. (34)] is replaced by the difference between Eqs. (34) and (33), i.e.,

$$\begin{aligned}&\bar\Delta C/\Delta\tau + \bar\Delta D/\Delta\tau + V^n\bar\Delta C_y + C_y^n\bar\Delta V + V^n\bar\Delta D_y + D_y^n\bar\Delta V \\ &= \tilde\Delta C/\Delta\tau + \tilde\Delta D/\Delta\tau\end{aligned} \tag{35}$$

It is to be pointed out that in Eqs. (33-35), as well as in the others not explicitly written, all of the apparent additional unknowns ($\Delta U, \Delta V, \Delta A$) at both sweeps are easily expressed by means of Eqs. (24-27) and (31) in terms of ΔC, ΔD, and ΔE at the corresponding sweeps.

As in the one-dimensional flow technique, all of the derivatives at the old time level t^n are approximated with second-order-accurate three-point windward differences, whereas simple first-order-accurate two-point windward differences are used for the derivatives of the incremental variables. In this way, only 3×3 block-tridiagonal systems have to be solved (by means of standard block-tridiagonal reduction[27,30]) at both sweeps of the ADI procedure. Again, it is noteworthy that, at the grid points immediately adjacent to the boundaries, standard central differences are used whenever it is not possible to employ three-point windward differences.

Boundary Conditions

The two-dimensional technique is presently limited to Cartesian coordinates, i.e., to the case of compressible flow past (symmetric) thin airfoils. An analytical stretching is used in both the horizontal and vertical directions, in order to allow for a better resolution of the flowfield for a given total number of grid points. Such a stretching introduces scale factors, $kx(x)$ and $ky(y)$, in all terms of the governing equations containing an x or y derivative, respectively. These metric coefficients, which transform the rectangular physical domain into a unit-square computational domain, have been omitted in all equations for the sake of conciseness. In such a computational domain, the following boundary conditions are required: inlet boundary conditions on the left side of the domain, outlet boundary conditions on the right side, far-field boundary conditions on the upper side, and symmetry or tangency conditions on the lower side. In all cases, the boundary conditions, together with the appropriate equations for the waves that reach the boundary under consideration (from within the computational domain), have to provide three relationships among the ΔC, ΔD, and ΔE unknowns at every grid point on the boundaries.

Far-Field Boundary Conditions

Values of C, D, and E are imposed corresponding to the freestream values.

Inlet Boundary Conditions

The inlet discontinuity surfaces proposed by Pandolfi[31] are used. The transversal component of the velocity V is assumed to be zero, i.e.,

$$-C+D+2E=0 \tag{36}$$

Moreover, the total temperature is imposed to be equal to its freestream value; such a condition, in conjunction with Eq. (36), is given by

$$A^2+U^2(\gamma-1)/2=A_0^2+U_0^2(\gamma-1)/2 \tag{37}$$

where U_0 and A_0 are the freestream speeds of flow and sound, respectively. Finally, the equation of propagation along the D bicharacteristic is used, which due to Eq. (36) becomes

$$D_\tau+(U-A)D_x=0 \tag{38}$$

Notice that Eq. (38) is valid only for subsonic entrance, which is the only case considered here, the interest of the present paper being the calculation of transonic flows.

Outlet Boundary Conditions

Downstream of the airfoil the pressure disturbances produced by the (thin) airfoil are assumed to decay according to small-perturbation theory. After linearization, the following equation results:

$$x_I A_I - x_{I-1} A_{I-1} = A_0 (x_I - x_{I-1}) \tag{39}$$

where x_I is the abscissa of the outlet boundary grid point. Furthermore, the propagation equation of the C wave [Eq. (20)] is used together with the governing equation (29).

Symmetry and Tangency Conditions

Along the symmetry line or at the airfoil surface, the symmetry or tangency condition is imposed

$$V/U=\tan\theta \tag{40}$$

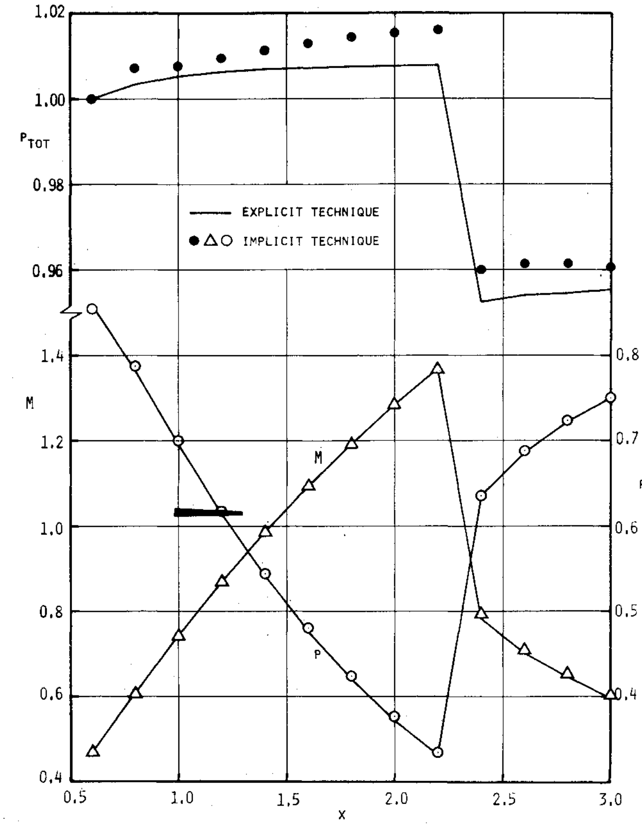


Fig. 1a Pressure, total pressure, and Mach number distributions (12 meshes).

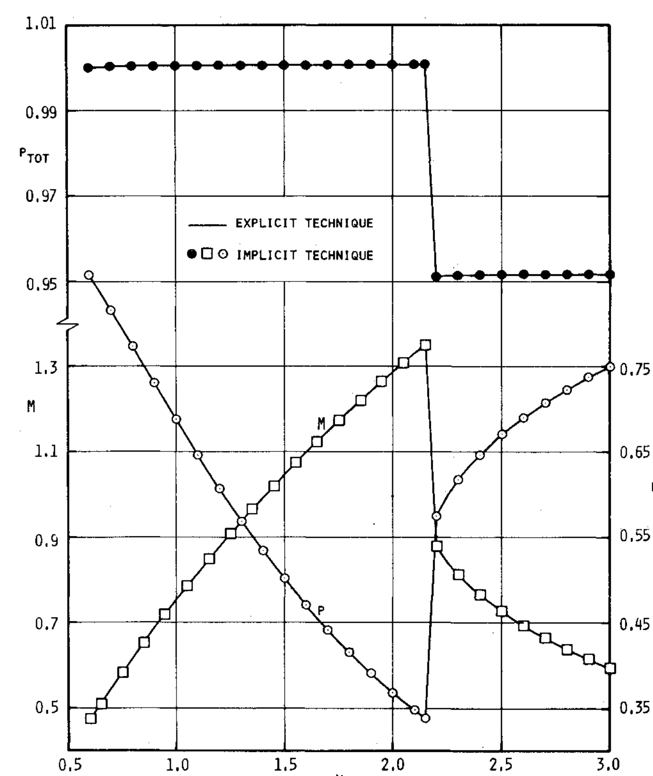
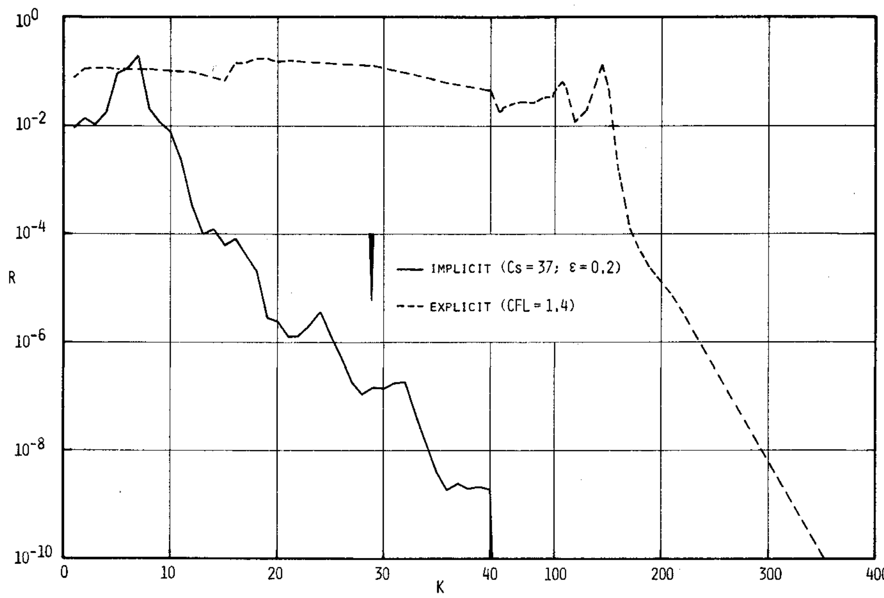


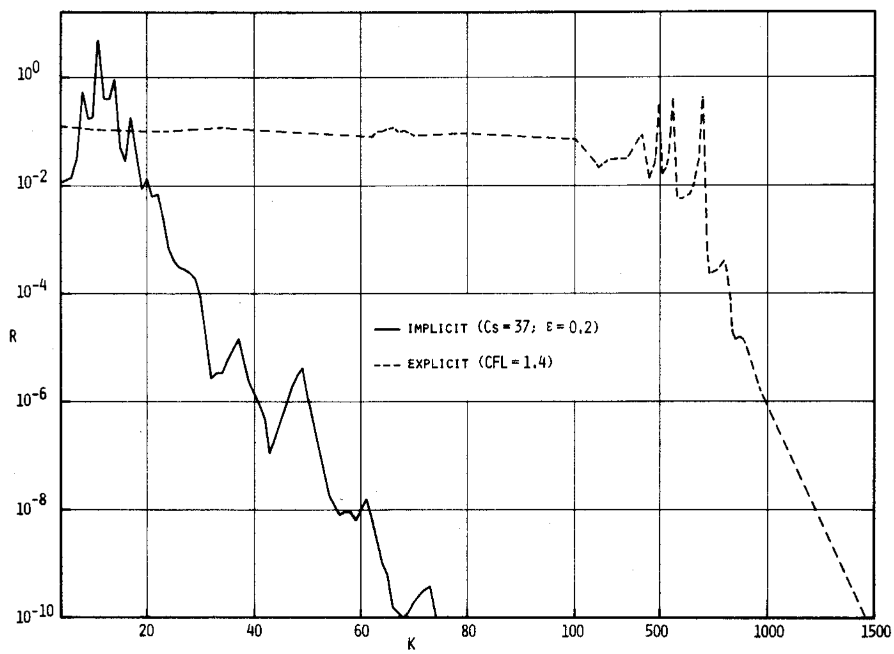
Fig. 1b Pressure, total pressure, and Mach number distributions (48 meshes).

where $\tan\theta$ is either the slope of the airfoil (upper) surface or zero. The governing equation (28) is also used, together with the following relationship:

$$2C_\tau - 2D_\tau - 2E_\tau = -\{(U+A)C_x - (U-A)D_x - U(E_x+F_x) - 2(V-A)F_y\} \tag{41}$$



a)



b)

Fig. 2 Convergence history: a) 12 meshes, b) 48 meshes.

This is obtained in a way similar to Eq. (30) by taking into account that there is no E wave coming from the body surface.

Results

One-Dimensional Flows

The present technique has been applied to two test problems. The first is the one-dimensional transonic flow inside a convergent-divergent nozzle whose nondimensional cross-sectional area W is given by

$$W = x/2 + 1/x \quad (0.6 \leq x \leq 3) \tag{42}$$

Starting from rest, the downstream static pressure has been reduced, within a few time steps, to its final value equal to 0.75 times the inlet total pressure. In Fig. 1, the steady-state Mach number and static and total pressure distributions along the nozzle are given for the case of the present implicit method, as well as for the case of the classical explicit one,[20] using 12 and 48 computational meshes. The agreement between the two methods is satisfactory for the coarse mesh and perfect for the finer one. The minor differences are most likely due to the different treatment of the boundary conditions, which in the explicit technique are imposed as in Ref. 18. Obviously, when the mesh is refined the two methods provide identical results.

Since the main advantage of the present approach is its presumed computational efficiency, a comparison between the convergence rates of the two methods is provided in Fig. 2, by plotting the maximum value of R ($R = |(\Delta z/\Delta \tau)/z|$, z being either C or D) vs the iteration step number K. From Fig. 2, it appears that the present implicit technique is about 10 times faster than the explicit method, which is taken from Ref. 18 and therefore allows for a CFL number greater than one. It is noteworthy to point out that, after the flow has become supersonic at one grid point, a time step cyclic variation according to Ref. 26 has been found beneficial to enhance the convergence rate of the method.

If one considers that the explicit approach is a two-step predictor-corrector technique and that the implicit method is a single-step procedure, which requires the solution of a 2×2 block-tridiagonal system per time step (so that the computer time per time-step is about the same for the two approaches), the superiority of the present implicit method is self-evident.

Besides such a considerable efficiency gain in reaching a steady-state solution, it is a claim of the authors that the present approach is also superior to its explicit counterpart for true time-dependent problems, for which the stability limitation on the time step of explicit techniques is often much more restrictive than that required for an accurate representation of a transient phenomenon. Such a point has been pursued by means of two examples. For the transient phenomenon of the nozzle flow, already considered, the Mach number history at a supersonic grid point has been given in Fig. 3 for both the explicit method (for which the time step is restricted by the CFL condition) and the implicit method for various combinations of two parameters Cs and ϵ. Cs is the ratio between the time step $\Delta\tau$ and the mesh width Δx; ϵ is an accuracy limitation that is to be satisfied at every time step by the maximum value of $|\Delta z/z|$ (z being either C or D). In case such a limitation is violated, the entire time step calculation is repeated with a smaller $\Delta\tau$, such that the aforementioned accuracy requirement is satisfied. Figure 3 clearly indicates that, except possibly for the shock motion, an accurate transient is also obtained for values of Cs and ϵ such that the entire calculation process uses an average $\Delta\tau$ corresponding to a CFL number of about 15. The second example considered here has been chosen to further pursue such a point: for a constant-area duct having constant inlet conditions, the downstream pressure has been allowed to vary sinusoidally with time. The computed Mach number at the outlet section is given in Fig. 4 for both the explicit and implicit methods, again for various values of Cs and ϵ. The results clearly indicate that an accurate unsteady phenomenon can be obtained with a computation time up to 30 times smaller than that required by the classical explicit method.[20]

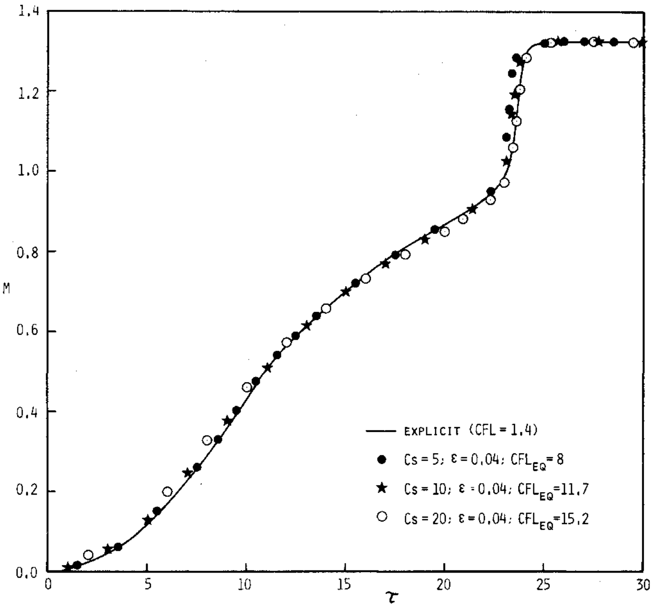


Fig. 3 Mach number vs time (24 meshes).

It is noteworthy to point out that, although a formal stability analysis has not been made, the present technique has been verified to be unconditionally stable. However, it is convenient to limit the time step whenever a shock is moving inside the flowfield for two main reasons. First, because the present implicit lambda scheme, like its explicit predecessor, is not capable of allowing shocks to move upstream inside a supersonic region[32]; therefore, good accuracy is required during the transient in order to prevent the shock moving downstream of its correct final location. Second, for efficiency purposes, in order to avoid the introduction of large errors that require costly computer time to be removed via relaxation. The aforementioned limitation of the time step is automatically performed by means of the parameter ϵ.

Finally, it is noteworthy that the superiority of the present method with respect to its explicit predecessor, as a true time-dependent solver, is very likely to be further enhanced by improving its accuracy in time to second order.

Two-Dimensional Flows

The present two-dimensional ADI technique has been applied to the computation of the transonic flows past two symmetric circular-arc airfoils having maximum thicknesses equal to 6 and 10% of the chord. Freestream Mach numbers of 0.91 and 0.83 have been considered for the thinner and thicker airfoils, respectively. These two examples have been

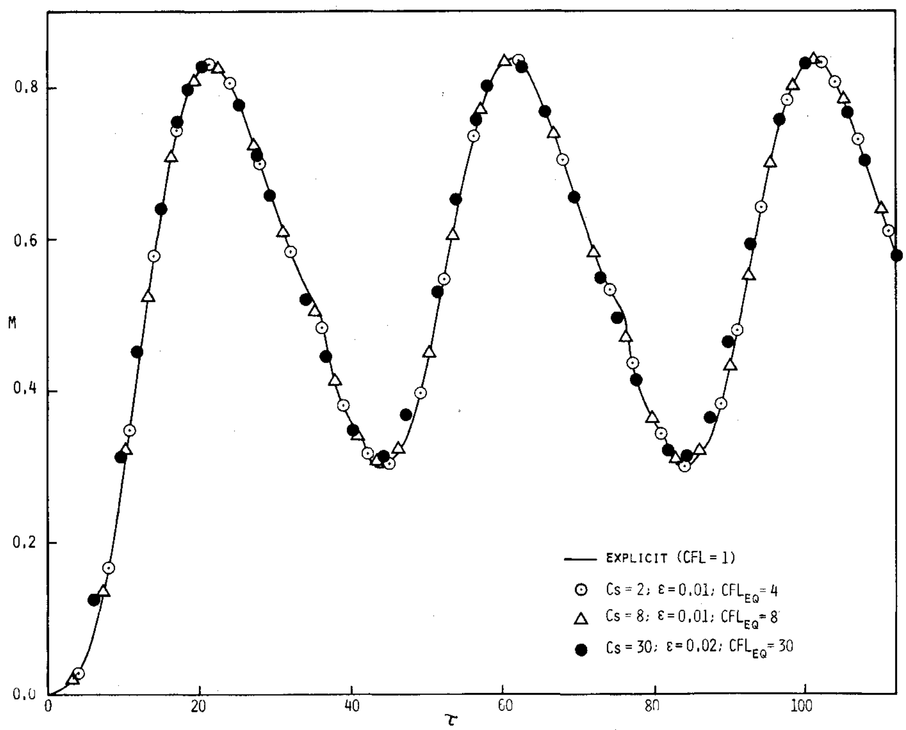


Fig. 4 Outlet Mach number vs time for a duct flow.

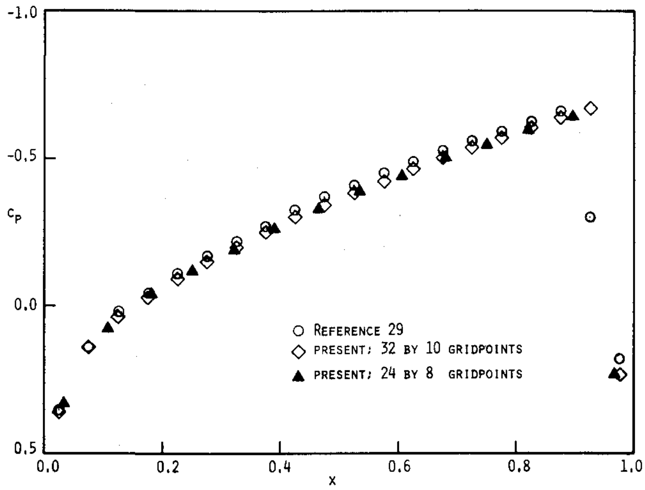


Fig. 5 Pressure coefficient distribution at the airfoil surface ($t/c = 0.06$).

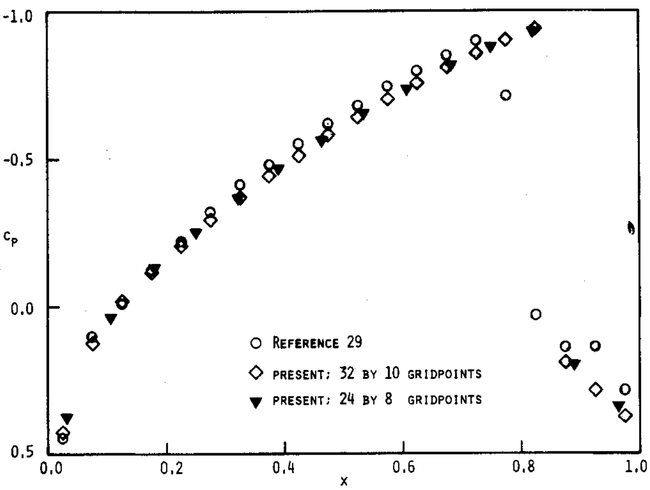


Fig. 6 Pressure coefficient distribution at the airfoil surface ($t/c = 0.1$).

taken from the literature in order to compare the present findings with previous results.[29] Figures 5 and 6 provide the present steady-state pressure coefficient results at the airfoil surface (i.e., on the horizontal axis—insofar as thin airfoil theory has been used to transfer the surface boundary condition onto the horizontal axis) compared with those of Beam and Warming.[29] The agreement is satisfactory and verifies the correctness of the present approach. The results given in Figs. 5 and 6 have been obtained using 32×10 and 24×8 grid points in the longitudinal and vertical directions, respectively, for both cases. An analytical stretching has been used in both directions, such that in the physical plane the vertical mesh size grows at a constant rate of 1.2 and the horizontal mesh size is constant on the surface of the airfoil and grows at a constant rate of 1.5 outside of it. The boundary conditions are imposed at a distance of 0.64, 1.6, and 2.6 chord lengths (0.55, 1.46, and 2.46 for the coarser mesh) ahead of, behind, and above the airfoils, respectively.

Two points are of particular interest: 1) the shock is implicitly captured by the technique within one mesh, a well-known capability of the classical lambda scheme; 2) no artificial viscosity nor higher order dissipation, necessary for the calculations of Beam and Warming,[29] has been used to obtain the present results.

In order to assess the efficiency of the present ADI method, the maximum value of R ($R = |(\Delta z/\Delta\tau)/z|$, z being any of the C, D, E, and F variables) is plotted vs the iteration step number K in Figs. 7 and 8 for the two airfoils and for both meshes. It is seen that a satisfactory convergence ($R = 10^{-4}$) is obtained within 60-80 ADI iterations in all cases. The results presented in Figs. 7 and 8 have been obtained for $Cs = 4.5$ and $\epsilon = 0.03$. Here, Cs is the ratio between the time step $\Delta\tau$ and the minimum absolute value of $\Delta y/(V \pm A)$, i.e., it is a transversal CFL number; ϵ is again the maximum specific variation of any of the four dependent variables ($\epsilon = |\Delta z/z|$) allowed in the calculations. All of the calculations have been performed by starting with the correct freestream flow past a flat plate, which has then been allowed to grow linearly into the desired airfoil within a time interval of 2.5. For the present two-dimensional flow results, a one-to-one comparison between the implicit and explicit techniques was not possible, because an explicit code directly applicable for the present test problem was not readily available. However, it seems plausible to simulate the explicit technique by means of the implicit one, working with a time step corresponding to a unit

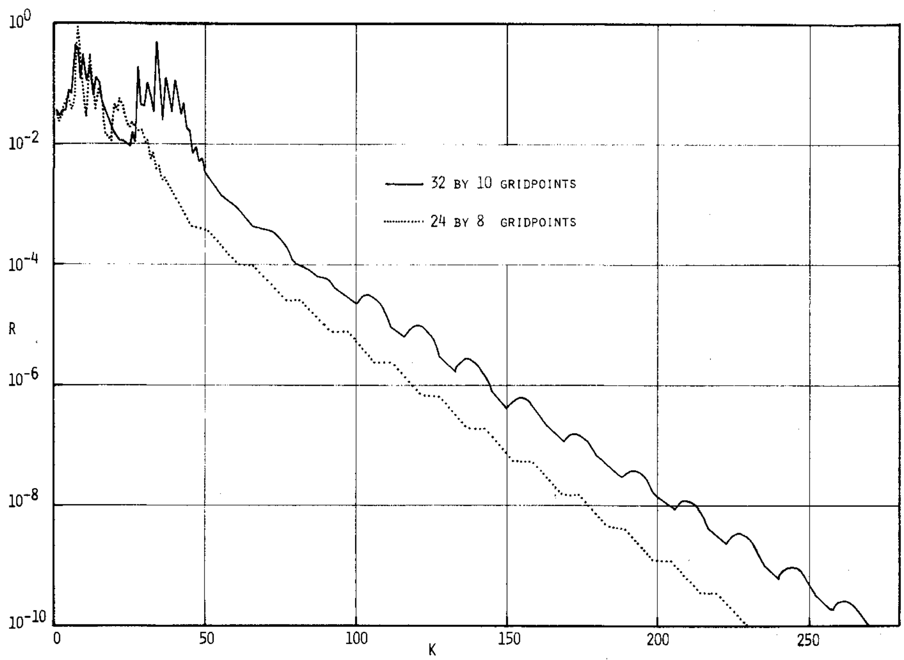


Fig. 7 Convergence history ($t/c = 0.06$).

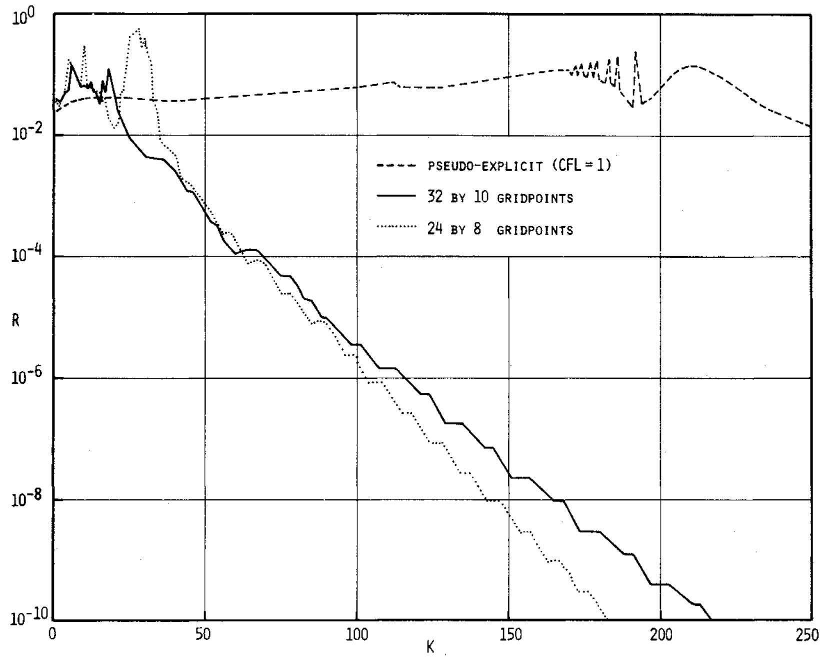


Fig. 8 Convergence history ($t/c = 0.1$).

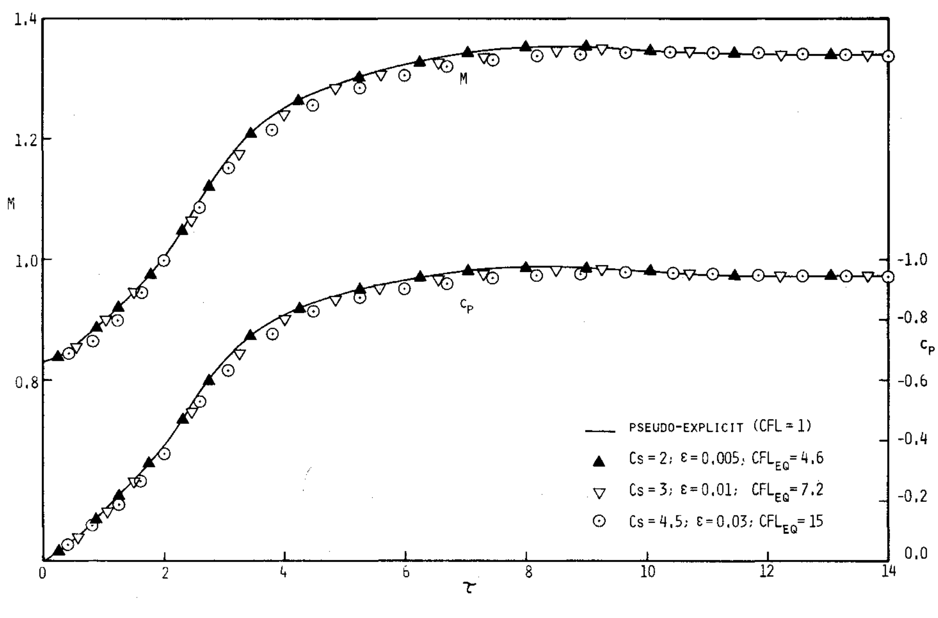


Fig. 9 Mach number and pressure coefficient vs time ($t/c = 0.1$).

CFL number. The convergence rate, obtained by means of such a pseudoexplicit method, for the case of the thicker airfoil and using the finer mesh is also plotted in Fig. 8. There it can be seen that after 250 iterations R is still of the order of 10^{-2}; satisfactory convergence ($R = 10^{-4}$) is reached only after about 1000 iterations.

The capability of the present two-dimensional ADI technique as a true time-dependent flow solver has also been assessed with respect to the aforementioned transient phenomenon produced by the thickening of the airfoil. The values of the pressure coefficient and of the Mach number at the last supersonic point on the (thicker) airfoil surface have been plotted vs time for the pseudoexplicit technique and for the implicit one for various values of Cs and ϵ in Fig. 9. There it can be seen that also the ADI technique, as the one-dimensional flow implicit one, can provide a reasonable resolution of an unsteady phenomenon with an average CFL number of 15. Obviously, since the present method is only first-order accurate in time, further improvements could be obtained by means of a second-order accurate technique.

Finally, it is to be pointed out that the present technique has a stability limitation. This has been numerically verified to correspond (for the present cases) to a transversal CFL number of about 4.5, corresponding to a CFL number of about 20. However, in spite of its conditional stability, the present technique is considered a major step toward obtaining a reliable and efficient solver of the time-dependent Euler equations.

Conclusions

An implicit technique based on the lambda scheme has been developed for one-dimensional compressible inviscid flows, together with its ADI extension to two-dimensional flows. Both methods have been shown to provide a considerable efficiency gain with respect to the classical explicit technique, while retaining its well-known "merits," as well as its minor drawbacks (see, e.g., Ref. 32). It is the authors' intention to extend the present ADI approach to arbitrary two-dimensional flows as well as to three-dimensional flows in order to better assess its capability to resolve flows of greater engineering interest.

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